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The influence of thermal radiation on MHD flow of a second grade fluid

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Abstract

The present analysis deals with the steady magnetohydrodynamic (MHD) flow of a second grade fluid in the presence of radiation. By means of similarity transformation, the arising non-linear partial differential equations are reduced to a system of four coupled ordinary differential equations. The series solutions of coupled system of equations are constructed for velocity and temperature using homotopy analysis method (HAM). Convergence of the obtained series solution is discussed. The effects of various involved interesting parameters on the velocity and temperature fields are shown and discussed.

Keywords: Boundary layer flow; Heat transfer analysis; Analytical solution

1. Introduction

Due to their application in industry and technology few problems in fluid mechanics have enjoyed the attention that has been accorded to the flow which involves non-Newtonian fluids. It is well known that mechanics of non-Newtonian fluids present a special challenge to engineers, physicists and mathematicians. The non-linearity can manifest itself in a variety of ways in many fields, such as food, drilling operations and bio-engineering. The Navier–Stokes theory is inadequate for such fluids and no single constitutive equation is available in the literature which exhibits the properties of all fluids. Because of complex behavior many fluid models have been suggested. Amongst these, the fluids of viscoelastic type have received much attention. In fact interest in viscoelastic fluids goes back almost to 65 years, triggered by the discovery of Mysels [1] and Toms [2] who found that the addition of small amounts of a high molecular weight polymer to a Newtonian fluid in turbulent pipe flows resulted in a dramatic decrease in pressure drop. The second grade fluid model is the simplest subclass of viscoelastic fluids for which one can reasonably hope to obtain the analytic solution. Some typical works on the topic are given in the references [3–12]. Even though considerable progress has been made in our understanding of the flow phenomena, more works are needed to understand the effects of the various parameters involved in the non-Newtonian models and the formulation of an accurate method of analysis for any body shapes of engineering significance. Also, the boundary layer concept for such fluids is of special importance because of its application to many practical problems, among which we cite the possibility of reducing frictional drag on the hulls of ships and submarines. Further, thermal radiation effects and MHD flow problems have assumed an increasing importance at a fundamental fabrication level. Specifically, such flows occurs in electrical power generation, astrophysical flows, solar power technology, space vehicle re-entry and other industrial areas [13,14]. Related studies regarding the thermal radiation of

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a gray fluid have been made in the references [15–20]. More recently, Raptis et al. [21] discussed the thermal radiation effects on the MHD flow of a viscous fluid.

The purpose of the present study is to examine the influence of thermal radiation on the MHD flow of a second grade fluid. The homotopy analysis method proposed by Liao [22,23] has been used for the analytic solution. HAM is recently developed powerful technique and has been successfully applied to several non-linear problems [25–44]. The organization of the paper is as follows:

In Section 2 the problem of MHD second grade fluid with radiation effects is formulated. Sections 3 and 4 comprise the series solutions for the flow and heat transfer analysis, respectively. The convergence of the solution is discussed in Section 5. The graphical results are presented and discussed in Section 6. Section 7 contains the concluding remarks.

2. Problem statement

Let us consider MHD flow of an incompressible second grade fluid past a semi-infinite fixed plate. We choose x-axis parallel and y-axis normal to the plate. A transverse magnetic field of strength B_0 is imposed. MHD equations are the usual electromagnetic and hydrodynamic equations, but modified to take account of the interaction between the motion and magnetic field. For small magnetic Reynolds number, the induced magnetic field is neglected. Moreover, the radiative heat flux in the x-direction is negligible when compared with the y-direction. For the present problem the conservation of mass and momentum equations can be expressed as [24]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+U\frac{\mathrm{d}U}{\mathrm{d}x}+\frac{\sigma B_0^2}{\rho}(U-u)+\frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right). \tag{2}$$

In the above equations $V=(u,v)$, U is the free stream velocity, α_1 is the material constant of second grade fluid, ρ and ν are the respective density and kinematic viscosity of fluid and σ is the electrical conductivity.

The energy equation is

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_\mathrm{p}}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho c_\mathrm{p}}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\alpha_1}{\rho c_\mathrm{p}}\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-\frac{1}{\rho c_\mathrm{p}}\frac{\partial q_\mathrm{r}}{\partial y}, \tag{3}$$

in which T is the fluid temperature, k is the thermal conductivity, c_p is the specific heat of the fluid under constant pressure and q_r is the radiative heat flux.

The appropriate boundary conditions are

$$u=0,\quad v=0,\quad T=T_w\quad \text{at } y=0,\qquad u\to U(x),\quad T\to T_\infty\quad \text{as } y\to\infty. \tag{4}$$

In above conditions T_w is the temperature at the plate, T_∞ is the temperature of the fluid far away from the plate and the free stream velocity is

$$U(x)=ax+cx^2, \tag{5}$$

in which a and c are constants.

Making use of the Rosseland approximation for radiation for an optically thick layer [15] one obtains

$$q_\mathrm{r}=-\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{6}$$

where k^* is the mean absorption coefficient and σ^* is the Stefan–Boltzmann constant. We express the term T^4 as a linear function of temperature. It is recognized by expanding T^4 in a Taylor series about T_∞ and neglecting higher terms, thus

$$T^4\cong 4T_\infty^3T-3T_\infty^4. \tag{7}$$

With the help of Eqs. (3), (6) and (7) we can write

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_\mathrm{p}}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho c_\mathrm{p}}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\alpha_1}{\rho c_\mathrm{p}}\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\frac{16\sigma^*T_\infty^3}{3k^*\rho c_\mathrm{p}}\frac{\partial^2 T}{\partial y^2}. \tag{8}$$

Employing the following transformations

$$\eta=\sqrt{\frac{a}{\nu}}y,\quad u=axf'(\eta)+cx^2g'(\eta),\qquad v=-\sqrt{a\nu}f(\eta)-\frac{2cx}{\sqrt{a/\nu}}g(\eta),\qquad T=T_w+(T_\infty-T_w)\left[T_0(\eta)+\frac{2c}{a}xT_1(\eta)\right]. \tag{9}$$

Eq. (1) is automatically satisfied and Eqs. (2), (4) and (8) yield

$$f'''-M^2f'-f'^2+ff''+\alpha\left(-f''^2+2f'f'''-ff''''\right)+(M^2+1)=0, \tag{10}$$

$$g'''-M^2g'-3g'f'+2gf''+g''f+(M^2+3)+\alpha(3g'''f'-3g''f''+3g'f'''-2gf''''-fg'''')=0, \tag{11}$$

$$(3K+4)T_0''+3KPfT_0'=0, \tag{12}$$

$$(3K+4)T_1''+3KP\left(-T_1f'+T_0'g+T_1'f\right)=0, \tag{13}$$

$$f(0)=0,\quad f'(0)=0,\quad f'(\infty)=1,\qquad g(0)=0,\quad g'(0)=0,\quad g'(\infty)=1,\qquad T_0(0)=0,\quad T_0(\infty)=1,\qquad T_1(0)=0,\quad T_1(\infty)=0. \tag{14}$$

In the above, the Prandtl number Pr, the radiation number K, the Hartman number M and the dimensionless material parameter α are defined, respectively, as:

$$Pr = \frac{\nu\rho c_p}{k}, \quad K = \frac{k^* k}{4\sigma^* T_\infty^3}, \quad M^2 = \frac{\sigma B_0^2}{a\rho}, \quad \alpha = \frac{a\alpha_1}{\mu}.$$

Note that the energy equations (12) and (13) in the present problem becomes similar to that of a viscous fluid case. It is further interesting to note that for $M = 1/\sqrt{\alpha}$ Eq. (10) has the following exact solution

$$f = \eta + \sqrt{\alpha}\left(e^{-\eta/\sqrt{\alpha}} - 1\right).$$

3. HAM solution for $f(\eta)$ and $g(\eta)$

In this section we employ the homotopy analysis method to solve Eqs. (10)–(14). For that we select

$$f_0(\eta) = \eta - 1 + e^{-\eta}, \tag{15}$$
$$g_0(\eta) = \eta - 1 + e^{-\eta}, \tag{16}$$

as the initial guess approximations for $f(\eta)$ and $g(\eta)$, respectively and

$$\mathscr{L}_1(f) = f''' + f'', \tag{17}$$

as the auxiliary linear operator which has the property

$$\mathscr{L}_1[C_1\eta + C_2 + C_3 e^{-\eta}] = 0, \tag{18}$$

where C_1, C_2 and C_3 are arbitrary constants.

We construct the zeroth order deformation problems as

$$(1-p)\mathscr{L}_1\left[\hat{f}(\eta,p) - f_0(\eta)\right] = p\hbar_1\mathscr{N}_1\left[\hat{f}(\eta,p)\right], \tag{19}$$
$$(1-p)\mathscr{L}_1[\hat{g}(\eta,p) - g_0(\eta)] = p\hbar_2\mathscr{N}_2\left[\hat{f}(\eta,p),\hat{g}(\eta,p)\right], \tag{20}$$
$$\hat{f}(0,p) = 0, \quad \hat{f}'(0,p) = 0, \quad \hat{f}'(\infty,p) = 1, \tag{21}$$
$$\hat{g}(0,p) = 0, \quad \hat{g}'(0,p) = 0, \quad \hat{g}'(\infty,p) = 1, \tag{22}$$

where $\hbar_1$ and $\hbar_2$ are non-zero auxiliary parameters, $p \in [0,1]$ is the embedding parameter and the non-linear differential operators $\mathscr{N}_1$ and $\mathscr{N}_2$ are defined by

$$\begin{aligned}\mathscr{N}_1[\hat{f}(\eta,p)] = {} & \frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3} - M^2\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + (1-\chi_m)(M^2+1) \\ & - \left(\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\right)^2 + \hat{f}(\eta,p)\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2} \\ & + \alpha\left\{-\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^2 + 2\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3} \right. \\ & \left. -\hat{f}(\eta,p)\frac{\partial^4\hat{f}(\eta,p)}{\partial\eta^4}\right\},\end{aligned} \tag{23}$$

$$\begin{aligned}&\mathscr{N}_2[\hat{f}(\eta,p),\hat{g}(\eta,p)] \\ &= \frac{\partial^3\hat{g}(\eta,p)}{\partial\eta^3} - M^2\frac{\partial\hat{g}(\eta,p)}{\partial\eta} + (1-\chi_m)(M^2+3) \\ &\quad - 3\frac{\partial\hat{g}(\eta,p)}{\partial\eta}\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + 2\hat{g}(\eta,p)\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2} + \hat{f}(\eta,p)\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2} \\ &\quad + \alpha\left\{3\frac{\partial^3\hat{g}(\eta,p)}{\partial\eta^3}\frac{\partial\hat{f}(\eta,p)}{\partial\eta} - 3\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2}\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right. \\ &\quad \left. +3\frac{\partial\hat{g}(\eta,p)}{\partial\eta}\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3} - 2\hat{g}(\eta,p)\frac{\partial^4\hat{f}(\eta,p)}{\partial\eta^4} - \frac{\partial^4\hat{g}(\eta,p)}{\partial\eta^{24}}\hat{f}(\eta,p)\right\}.\end{aligned} \tag{24}$$

Obviously for $p = 0$ and $p = 1$ we have

$$\hat{f}(\eta,0) = f_0(\eta), \quad \hat{f}(\eta,1) = f(\eta), \tag{25}$$
$$\hat{g}(\eta,0) = g_0(\eta), \quad \hat{g}(\eta,1) = g(\eta). \tag{26}$$

As p increases from 0 to 1, $\hat{f}(\eta,p)$ and $\hat{g}(\eta,p)$ vary from $f_0(\eta)$ and $g_0(\eta)$ to the exact solutions $f(\eta)$ and $g(\eta)$. Due to Taylor's theorem and Eqs. (25) and (26), we can express that

$$\hat{f}(\eta,p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \tag{27}$$
$$\hat{g}(\eta,p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta)p^m, \tag{28}$$
$$f_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m\hat{f}(\eta,p)}{\partial p^m}\right|_{p=0}, \quad g_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m\hat{g}(\eta,p)}{\partial p^m}\right|_{p=0}, \tag{29}$$

where the convergence of the series in Eqs. (27) and (28) is dependent upon $\hbar_1$ and $\hbar_2$. Assume that $\hbar_1$ and $\hbar_2$ are selected such that the series in Eqs. (27) and (28) are convergent at $p = 1$, then due to Eqs. (25) and (26) one can write

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{30}$$
$$g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta). \tag{31}$$

Differentiating the zeroth order deformation equations (19) and (20) m times with respect to p, then dividing by $m!$, and finally setting $p = 0$ we get the following mth-order deformation problems

$$\mathscr{L}_1[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_1\mathscr{R}_{1m}(\eta), \tag{32}$$
$$\mathscr{L}_1[g_m(\eta) - \chi_m g_{m-1}(\eta)] = \hbar_2\mathscr{R}_{2m}(\eta), \tag{33}$$
$$f_m(0) = f'_m(0) = f'_m(\infty) = 0, \tag{34}$$
$$g_m(0) = g'_m(0) = g'_m(\infty) = 0, \tag{35}$$
$$\begin{aligned}\mathscr{R}_{1m}(\eta) = {} & f'''_{m-1}(\eta) - M^2 f'_{m-1} + (1-\chi_m)(M^2+1) \\ & + \sum_{k=0}^{m-1}\left[-f'_{m-1-k}f'_k + f_{m-1-k}f''_k \right. \\ & \left. +\alpha\left(-f''_{m-1-k}f''_k + 2f'_{m-1-k}f'''_k - f_{m-1-k}f''''_k\right)\right],\end{aligned} \tag{36}$$

$$\mathcal{R}_{2m}(\eta) = g'''_{m-1}(\eta) - M^2 g'_{m-1} + (1-\chi_m)(M^2+3) + \sum_{k=0}^{m-1}\Big[-3g'_{m-1-k}f'_k + 2g_{m-1-k}f''_k + g''_{m-1-k}f_k. + \alpha\big(3g'''_{m-1-k}f'_k - 3g''_{m-1-k}f''_k + 3g'_{m-1-k}f'''_k - 2g_{m-1-k}f''''_k - g''''_{m-1-k}f_k\big)\Big], \quad (37)$$

where

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \quad (38)$$

MATHEMATICA is used to solve the linear equations (32)–(35) up to first few order of approximations and it is found that f and g can be expressed as

$$f_m(\eta) = \sum_{n=0}^{m+1}\sum_{q=0}^{2(m+1-n)} a^q_{m,n}\eta^q e^{-n\eta}, \quad m \geqslant 0, \quad (39)$$

$$g_m(\eta) = \sum_{n=0}^{m+1}\sum_{q=0}^{2(m+1-n)} b^q_{m,n}\eta^q e^{-n\eta}, \quad m \geqslant 0. \quad (40)$$

Substituting Eqs. (39) and (40) into Eqs. (32) and (33) the recurrence formulae for the coefficients $a^q_{m,n}$ and $b^q_{m,n}$ of $f_m(\eta)$ and $g_m(\eta)$ are obtained, respectively, for $m \geqslant 1$, $0 \leqslant n \leqslant m+1$ and $0 \leqslant q \leqslant 2(m+1-n)$ as

$$a^0_{m,0} = \chi_m\chi_{2m+2}a^0_{m-1,0} - \sum_{q=0}^{2m}\Delta1^q_{m,1}\mu^q_{1,1} - \sum_{n=2}^{m+1}\left[(n-1)\Delta1^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2(m+1-n)}\Delta1^q_{m,n}\Big((n-1)\mu^q_{n,0} - \mu^q_{n,1}\Big)\right], \quad (41)$$

$$a^k_{m,0} = \chi_m\chi_{2m+2-k}a^k_{m-1,0}, \quad 1 \leqslant k \leqslant 2m+2, \quad (42)$$

$$a^0_{m,1} = \chi_m\chi_{2m}a^0_{m-1,1} + \sum_{q=0}^{2m}\Delta1^q_{m,1}\mu^q_{1,1} + \sum_{n=2}^{m+1}\left\{n\Delta1^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2(m+1-n)}\Delta1^q_{m,n}(n\mu^q_{n,0} - \mu^q_{n,1})\right\}, \quad (43)$$

$$a^k_{m,1} = \chi_m\chi_{2m-k}a^k_{m-1,1} + \sum_{q=k-1}^{2m}\Delta1^q_{m,1}\mu^q_{1,k}, \quad 1 \leqslant k \leqslant 2m, \quad (44)$$

$$a^k_{m,n} = \chi_m\chi_{2(m-n)-k+2}a^k_{m-1,n} + \sum_{q=k}^{2(m+1-n)}\Delta1^q_{m,n}\mu^q_{n,k}, \quad 2 \leqslant n \leqslant m+1, \quad 0 \leqslant k \leqslant 2(m+1-n), \quad (45)$$

$$b^0_{m,0} = \chi_m\chi_{2m+2}b^0_{m-1,0} - \sum_{q=0}^{2m}\Delta2^q_{m,1}\mu^q_{1,1} - \sum_{n=2}^{m+1}\left[(n-1)\Delta2^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2(m+1-n)}\Delta2^q_{m,n}((n-1)\mu^q_{n,0} - \mu^q_{n,1})\right], \quad (46)$$

$$b^k_{m,0} = \chi_m\chi_{2m+2-k}b^k_{m-1,0}, \quad 1 \leqslant k \leqslant 2m+2, \quad (47)$$

$$b^0_{m,1} = \chi_m\chi_{2m}b^0_{m-1,1} + \sum_{q=0}^{2m}\Delta2^q_{m,1}\mu^q_{1,1} + \sum_{n=2}^{m+1}\left\{n\Delta2^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2(m+1-n)}\Delta2^q_{m,n}\Big(n\mu^q_{n,0} - \mu^q_{n,1}\Big)\right\}, \quad (48)$$

$$b^k_{m,1} = \chi_m\chi_{2m-k}b^k_{m-1,1} + \sum_{q=k-1}^{2m}\Delta2^q_{m,1}\mu^q_{1,k}, \quad 1 \leqslant k \leqslant 2m, \quad (49)$$

$$b^k_{m,n} = \chi_m\chi_{2(m-n)-k+2}b^k_{m-1,n} + \sum_{q=k}^{2(m+1-n)}\Delta2^q_{m,n}\mu^q_{n,k}, \quad 2 \leqslant n \leqslant m+1, \; 0 \leqslant k \leqslant 2(m+1-n), \quad (50)$$

where

$$\mu^q_{1,k} = \frac{(q-k+2)q!}{k!}, \quad 0 \leqslant k \leqslant q+1, \quad q \geqslant 0, \quad (51)$$

$$\mu^q_{n,k} = \sum_{p=0}^{q-k}\frac{(q-k-p+1)q!}{k!n^{q-k-p+2}(n-1)^{p+1}}, \quad 0 \leqslant k \leqslant q, \; q \geqslant 0, \; n \geqslant 2, \quad (52)$$

$$\Delta1^q_{m,n} = \hbar_1\Big[\chi_{2(m-n)-q+2}(a3^q_{m-1,n} - M^2 a1^q_{m-1,n}) - (\delta1^q_{m,n} - \delta2^q_{m,n}) + \alpha(-\delta3^q_{m,n} + 2\delta4^q_{m,n} - \delta5^q_{m,n})\Big], \quad (53)$$

$$\Delta2^q_{m,n} = \hbar_2\Big[\chi_{2(m-n)-q+2}(b3^q_{m-1,n} - M^2 b1^q_{m-1,n}) - (3\delta6^q_{m,n} - 2\delta7^q_{m,n} - \delta8^q_{m,n}) + \alpha(3\delta9^q_{m,n} - 3\delta10^q_{m,n} + 3\delta11^q_{m,n} - 2\delta12^q_{m,n} - \delta13^q_{m,n})\Big], \quad (54)$$

and $\delta1^q_{m,n} - \delta13^q_{m,n}$, where $m \geqslant 1$, $0 \leqslant n \leqslant 3m+3$, $0 \leqslant q \leqslant 3m+3-n$ are

$$\delta1^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}}\sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a1^i_{k,j}a1^{q-i}_{m-1-k,n-j},$$

$$\delta2^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}}\sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a2^i_{k,j}a^{q-i}_{m-1-k,n-j},$$

$$\delta3^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}}\sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a2^i_{k,j}a2^{q-i}_{m-1-k,n-j},$$

$$\delta4^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}}\sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a3^i_{k,j}a1^{q-i}_{m-1-k,n-j},$$

$$\delta5^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}}\sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a4^i_{k,j}a^{q-i}_{m-1-k,n-j},$$

$$\delta 6^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a1^i_{k,j} b1^{q-i}_{m-1-k,n-j},$$

$$\delta 7^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a2^i_{k,j} b^{q-i}_{m-1-k,n-j},$$

$$\delta 8^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a1^i_{k,j} b2^{q-i}_{m-1-k,n-j},$$

$$\delta 9^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a1^i_{k,j} b3^{q-i}_{m-1-k,n-j},$$

$$\delta 10^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a2^i_{k,j} b2^{q-i}_{m-1-k,n-j},$$

$$\delta 11^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a3^i_{k,j} b1^{q-i}_{m-1-k,n-j},$$

$$\delta 12^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a4^i_{k,j} b^{q-i}_{m-1-k,n-j},$$

$$\delta 13^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a^i_{k,j} b4^{q-i}_{m-1-k,n-j}.$$

The values $a1^k_{m,n}$, $a2^k_{m,n}$, $a3^k_{m,n}$, $a4^k_{m,n}$, $b1^k_{m,n}$, $b2^k_{m,n}$, $b3^k_{m,n}$ and $b4^k_{m,n}$ are

$$\begin{aligned} a1^k_{m,n} &= (k+1)a^{k+1}_{m,n} - na^k_{m,n}, \\ a2^k_{m,n} &= (k+1)a1^{k+1}_{m,n} - na1^k_{m,n}, \\ a3^k_{m,n} &= (k+1)a2^{k+1}_{m,n} - na2^k_{m,n}, \\ a4^k_{m,n} &= (k+1)a3^{k+1}_{m,n} - na3^k_{m,n}, \end{aligned} \tag{55}$$

$$\begin{aligned} b1^k_{m,n} &= (k+1)b^{k+1}_{m,n} - nb^k_{m,n}, \\ b2^k_{m,n} &= (k+1)b1^{k+1}_{m,n} - nb1^k_{m,n}, \\ b3^k_{m,n} &= (k+1)b2^{k+1}_{m,n} - nb2^k_{m,n}, \\ b4^k_{m,n} &= (k+1)b3^{k+1}_{m,n} - nb3^k_{m,n}. \end{aligned} \tag{56}$$

In order to see the detailed procedure of deriving the above relations the reader may consult [25]. Using the above recurrence formulae, we can calculate all coefficients $a^k_{m,n}$ and $b^k_{m,n}$ using only the first four

$$\begin{aligned} &a^0_{0,0} = -1, \quad a^1_{0,0} = 1, \quad a^0_{0,1} = 1, \quad a^2_{0,0} = 0, \\ &b^0_{0,0} = -1, \quad b^1_{0,0} = 1, \quad b^0_{0,1} = 1, \quad b^2_{0,0} = 0 \end{aligned} \tag{57}$$

given by the initial guess approximation in Eqs. (15) and (16). The corresponding Mth-order approximation of Eqs. (10), (11) and (14) are

$$\sum_{m=0}^{M} f_m(\eta) = \sum_{m=0}^{M} a^0_{m,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1-n)} a^k_{m,n} \eta^k \right), \tag{58}$$

$$\sum_{m=0}^{M} g_m(\eta) = \sum_{m=0}^{M} b^0_{m,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1-n)} b^k_{m,n} \eta^k \right). \tag{59}$$

Therefore the following explicit, totally analytic solution of the present flow is

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) = \lim_{M\to\infty} \left[\sum_{m=0}^{M} a^0_{m,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1-n)} a^k_{m,n} \eta^k \right) \right], \tag{60}$$

$$g(\eta) = \sum_{m=0}^{\infty} g_m(\eta) = \lim_{M\to\infty} \left[\sum_{m=0}^{M} b^0_{m,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1-n)} b^k_{m,n} \eta^k \right) \right]. \tag{61}$$

4. HAM solution for $T_0(\eta)$ and $T_1(\eta)$

The initial guess approximations for $T_0(\eta)$ and $T_1(\eta)$ are

$$T^0_0(\eta) = 1 - e^{-\eta}, \tag{62}$$

$$T^0_1(\eta) = \eta e^{-\eta}, \tag{63}$$

and the auxiliary operators are

$$\mathscr{L}_2 = f'' + f', \tag{64}$$

$$\mathscr{L}_3 = f'' + 2f' + f, \tag{65}$$

which satisfy the properties

$$\mathscr{L}_2[C_5 + C_4 e^{-\eta}] = 0, \tag{66}$$

$$\mathscr{L}_3[(C_7 + C_6\eta)e^{-\eta}] = 0, \tag{67}$$

in which C_4, C_5, C_6 and C_7 are arbitrary constants. The zeroth order deformation problems are

$$(1-p)\mathscr{L}_2[\widehat{T}_0(\eta,p) - T^0_0(\eta)] = p\hbar_3 \mathscr{N}_3[\widehat{T}_0(\eta,p), \hat{f}(\eta,p)], \tag{68}$$

$$(1-p)\mathscr{L}_3[\widehat{T}_1(\eta,p) - T^0_1(\eta)] = p\hbar_4 \mathscr{N}_4[\widehat{T}_0(\eta,p), \widehat{T}_1(\eta,p), \hat{f}(\eta,p), \hat{g}(\eta,p)], \tag{69}$$

$$\widehat{T}_0(0,p)=0,\quad \widehat{T}_0(\infty,p)=1,$$
$$\widehat{T}_1(0,p)=0,\quad \widehat{T}_1(\infty,p)=0, \tag{70}$$
$$\mathcal{N}_3\left[\widehat{T}_0(\eta,p),\hat{f}(\eta,p)\right]=\frac{\partial^2\widehat{T}_0(\eta,p)}{\partial\eta^2}+\frac{3KP}{(3K+4)}\hat{f}(\eta,p)\frac{\partial\widehat{T}_0(\eta,p)}{\partial\eta}, \tag{71}$$
$$\mathcal{N}_4\left[\widehat{T}_0(\eta,p),\widehat{T}_1(\eta,p),\hat{f}(\eta,p),\hat{g}(\eta,p)\right]=\frac{\partial^2\widehat{T}_1(\eta,p)}{\partial\eta^2}+\frac{3KP}{(3K+4)}\left(-\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\widehat{T}_1(\eta,p)+\hat{g}(\eta,p)\frac{\partial\widehat{T}_0(\eta,p)}{\partial\eta}+\hat{f}(\eta,p)\frac{\partial\widehat{T}_1(\eta,p)}{\partial\eta}\right). \tag{72}$$

In Eqs. (68) and (69), $\hbar_3$ and $\hbar_4$ are the auxiliary nonzero parameters. For $p=0$ and $p=1$ we have

$$\widehat{T}_0(\eta,0)=T_0^0(\eta),\quad \widehat{T}_0(\eta,1)=T_0(\eta), \tag{73}$$
$$\widehat{T}_1(\eta,0)=T_1^0(\eta),\quad \widehat{T}_1(\eta,1)=T_1(\eta). \tag{74}$$

As p increases from 0 to 1, $\widehat{T}_0(\eta,p)$ and $\widehat{T}_1(\eta,p)$ vary from the initial guesses $T_0^0(\eta)$ and $T_1^0(\eta)$ to the exact solutions $T_0(\eta)$ and $T_1(\eta)$ respectively. By Taylor's theorem and Eqs. (68) and (69), one obtains

$$\widehat{T}_0(\eta,p)=T_0^0(\eta)+\sum_{m=1}^{\infty}T_0^m(\eta)p^m, \tag{75}$$
$$\widehat{T}_1(\eta,p)=T_1^0(\eta)+\sum_{m=1}^{\infty}T_1^m(\eta)p^m, \tag{76}$$
$$T_0^m(\eta)=\frac{1}{m!}\left.\frac{\partial^m\widehat{T}_0(\eta,p)}{\partial p^m}\right|_{p=0},$$
$$T_1^m(\eta)=\frac{1}{m!}\left.\frac{\partial^m\widehat{T}_1(\eta,p)}{\partial p^m}\right|_{p=0}. \tag{77}$$

Clearly the convergence of the series (75) and (76) strongly depends upon $\hbar_3$ and $\hbar_4$. The values of $\hbar_3$ and $\hbar_4$ are selected in such a way that the series (75) and (76) are convergent at $p=1$, then due to Eqs. (73) and (74) we have

$$T_0(\eta)=T_0^0(\eta)+\sum_{m=1}^{\infty}T_0^m(\eta), \tag{78}$$
$$T_1(\eta)=T_1^0(\eta)+\sum_{m=1}^{\infty}T_1^m(\eta). \tag{79}$$

For mth-order deformation problems, we employ a similar procedure as in previous section and obtain

$$\mathcal{L}_2\left[T_0^m(\eta)-\chi_mT_0^{m-1}(\eta)\right]=\hbar_3\mathcal{R}_{3m}(\eta), \tag{80}$$
$$\mathcal{L}_3\left[T_1^m(\eta)-\chi_mT_1^{m-1}(\eta)\right]=\hbar_4\mathcal{R}_{4m}(\eta), \tag{81}$$
$$T_0^m(0)=T_0^m(\infty)=0,\ T_1^m(0)=T_1^m(\infty)=0, \tag{82}$$
$$\mathcal{R}_{3m}(\eta)=\frac{\partial^2T_0^{m-1}(\eta)}{\partial\eta^2}+\frac{3KP}{(3K+4)}\sum_{k=0}^{m-1}\frac{\partial T_0^{m-1-k}}{\partial\eta}f_k, \tag{83}$$
$$\mathcal{R}_{4m}(\eta)=\frac{\partial^2T_1^{m-1}(\eta)}{\partial\eta^2}+\frac{3KP}{(3K+4)}\times\sum_{k=0}^{m-1}\left(-T_1^{m-1-k}f_k'+\frac{\partial T_1^{m-1-k}}{\partial\eta}f_k+\frac{\partial T_0^{m-1-k}}{\partial\eta}g_k\right). \tag{84}$$

The solutions of Eqs. (80)–(82) is

$$T_0^m(\eta)=\sum_{n=0}^{m+1}\sum_{q=0}^{2(m+1-n)}c_{m,n}^q\eta^q e^{-n\eta},\quad m\geqslant 0, \tag{85}$$
$$T_1^m(\eta)=\sum_{n=0}^{m+1}\sum_{q=0}^{2m+6-2n}d_{m,n}^q\eta^q e^{-n\eta},\quad m\geqslant 0. \tag{86}$$

Using Eqs. (85) and (86) into Eqs. (80) and (81), the recurrence formulae for the coefficients $c_{m,n}^q$ of $T_0^m(\eta)$ for $m\geqslant 1$, $0\leqslant n\leqslant m+1$, and $0\leqslant q\leqslant 2m+2-2n$ and $d_{m,n}^q$ of $T_1^m(\eta)$ for $m\geqslant 1$, $0\leqslant n\leqslant m+1$, and $0\leqslant q\leqslant 2m+6-2n$ are obtained as

$$c_{m,0}^k=\chi_m\chi_{2m+2-k}c_{m-1,0}^k,\quad 0\leqslant k\leqslant 2m+2, \tag{87}$$
$$c_{m,1}^0=\chi_m\chi_{2m}c_{m-1,1}^0-\sum_{n=2}^{m+1}\sum_{q=0}^{2(m+1-n)}\Delta 3_{m,n}^q\mu 1_{n,0}^q, \tag{88}$$
$$c_{m,1}^k=\chi_m\chi_{2m-k}c_{m-1,1}^k-\sum_{q=k-1}^{2m}\Delta 3_{m,1}^q\mu 1_{1,k}^q,\quad 1\leqslant k\leqslant 2m, \tag{89}$$
$$c_{m,n}^k=\chi_m\chi_{2(m-n)+2-k}c_{m-1,n}^k-\sum_{q=k}^{2(m+1-n)}\Delta 3_{m,n}^q\mu 1_{n,k}^q,\quad 2\leqslant n\leqslant m+1,\ 0\leqslant k\leqslant 2(m+1-n), \tag{90}$$
$$d_{m,1}^0=\chi_m\chi_{2m+4}d_{m-1,1}^0-\sum_{n=2}^{m+1}\sum_{q=0}^{2m+6-2n}\Delta 4_{m,n}^q\mu 2_{n,0}^q, \tag{91}$$
$$d_{m,1}^k=\chi_m\chi_{2m-k+4}d_{m-1,1}^k+\sum_{q=k-1}^{2m+4}\Delta 4_{m,1}^q\mu 2_{1,k}^q,\quad 1\leqslant k\leqslant 2m+4, \tag{92}$$
$$d_{m,n}^k=\chi_m\chi_{2m-2n-k+6}d_{m-1,n}^k+\sum_{q=k}^{2m+6-2n}\Delta 4_{m,n}^q\mu 2_{n,k}^q,\quad 2\leqslant n\leqslant m+1,\ 0\leqslant k\leqslant 2m+6-2n, \tag{93}$$

where

$$\mu 1_{1,k}^q=\frac{q!}{k!},\quad 0\leqslant k\leqslant q+1,\ q\geqslant 0, \tag{94}$$
$$\mu 1_{n,k}^q=\sum_{p=0}^{q-k}\frac{q!}{k!n^{p+1}(n-1)^{q-p+1}},\quad 0\leqslant k\leqslant q,\ q\geqslant 0,\ n\geqslant 2, \tag{95}$$
$$\mu 2_{1,k}^q=\frac{1}{(q+1)(q+2)},\quad q\geqslant 0, \tag{96}$$

$$\mu 2^q_{n,k} = \frac{(q-k+1)q!}{k!(n-1)^{q-p+2}}, \quad 0 \leqslant k \leqslant q, \quad q \geqslant 0, \ n \geqslant 2, \quad (97)$$

$$\Delta 3^q_{m,n} = \hbar_3 \left[\chi_{2(m-n)-q+2} c2^q_{m-1,n} + \frac{3KP}{(3K+4)} \delta 14^q_{m,n} \right], \quad (98)$$

$$\Delta 4^q_{m,n} = \hbar_4 \Bigg[\chi_{2m+6-2n-q} d2^q_{m-1,n} + \frac{3KP}{(3K+4)} \times \left(\chi_{2m+4-2n-q}(-\delta 15^q_{m,n} + \delta 16^q_{m,n}) + \delta 17^q_{m,n} \right) \Bigg], \quad (99)$$

and the coefficients $\delta 14^q_{m,n}$, $\delta 15^q_{m,n}$, $\delta 16^q_{m,n}$, and $\delta 17^q_{m,n}$, where $m \geqslant 1$, $0 \leqslant n \leqslant m+1$, $0 \leqslant q \leqslant 2(m+1-n)$ and $0 \leqslant q \leqslant 2m+6-2n$ are

$$\delta 14^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times a^i_{k,j} c1^{q-i}_{m-1-k,n-j},$$

$$\delta 15^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2m+2k-4+2n-2j\}}^{\min\{q,2(k+2-2j)\}} \times a1^i_{k,j} d^{q-i}_{m-1-k,n-j},$$

$$\delta 16^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2m+2k-4+2n-2j\}}^{\min\{q,2(k+2-2j)\}} \times a^i_{k,j} d1^{q-i}_{m-1-k,n-j},$$

$$\delta 17^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k+1-j)\}} \times b^i_{k,j} c1^{q-i}_{m-1-k,n-j},$$

where

$$\begin{aligned} c1^k_{m,n} &= (k+1)c^{k+1}_{m,n} - nc^k_{m,n}, \\ c2^k_{m,n} &= (k+1)ac1^{k+1}_{m,n} - ca1^k_{m,n}, \end{aligned} \quad (100)$$

$$\begin{aligned} d1^k_{m,n} &= (k+1)d^{k+1}_{m,n} - nd^k_{m,n}, \\ d2^k_{m,n} &= (k+1)d1^{k+1}_{m,n} - nd1^k_{m,n}. \end{aligned} \quad (101)$$

Using the same procedure as in Section 3, we can calculate all coefficients $c^k_{m,n}$ and $d^k_{m,n}$ using only the first few

$$\begin{aligned} &c^0_{0,0} = 1, \quad c^0_{0,1} = -1, \quad c^1_{0,0} = c^2_{0,0} = 0, \\ &d^1_{0,1} = 0, \quad d^0_{0,0} = b^{-1}_{0,0} = b^0_{0,1} = b^{-2}_{0,0} = 0, \end{aligned} \quad (102)$$

given by the initial guess approximation in Eqs. (62) and (63). The corresponding Mth-order approximation of Eqs. (12)–(14) are

$$\sum_{m=0}^{M} T^m_0(\eta) = \sum_{m=0}^{M} c^0_{m,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1-n)} c^k_{m,n} \eta^k \right), \quad (103)$$

$$\sum_{m=0}^{M} T^m_1(\eta) = \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2m+6-2n} d^k_{m,n} \eta^k \right). \quad (104)$$

and thus

$$T_0(\eta) = \sum_{m=0}^{M} T^m_0(\eta) = \lim_{M\to\infty} \left[\sum_{m=0}^{M} c^0_{m,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1-n)} c^k_{m,n} \eta^k \right) \right], \quad (105)$$

$$T_1(\eta) = \sum_{m=0}^{M} T^m_1(\eta) = \lim_{M\to\infty} \left[\sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2m+6-2n} d^k_{m,n} \eta^k \right) \right]. \quad (106)$$

5. Convergence of the HAM solution

As pointed out by Liao [22], the convergence and rate of approximation for the HAM solution strongly depends on the values of auxiliary parameters $\hbar_1$, $\hbar_2$, $\hbar_3$ and $\hbar_4$. To see the admissible values of $\hbar_1$, $\hbar_2$, $\hbar_3$ and $\hbar_4$, $\hbar$-curves are plotted for two different orders of approximations. Figs. 1–4 clearly depict that the range for the admissible values for $\hbar_1$, $\hbar_2$, $\hbar_3$ and $\hbar_4$ is $-0.04 \leqslant \hbar_1 < 0$, $-0.15 \leqslant \hbar_2 < 0$,

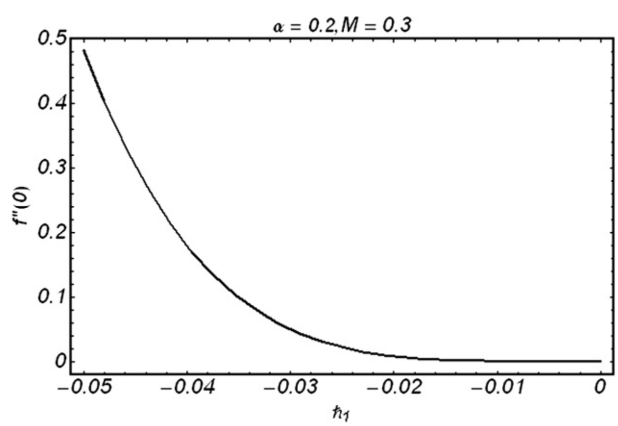

Fig. 1. $\hbar_1$-curve for the 8th-order of approximation.

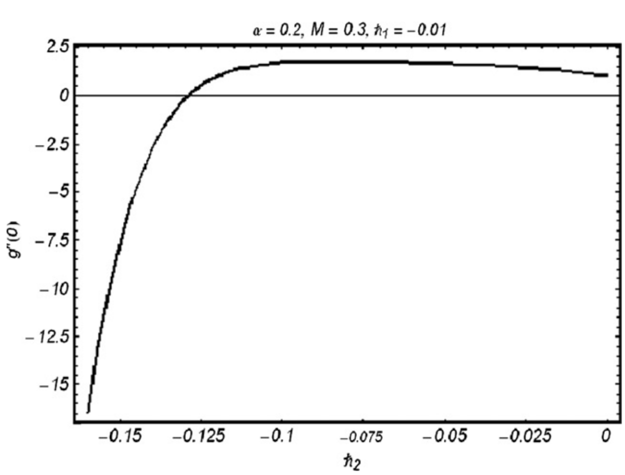

Fig. 2. $\hbar_2$-curve for the 8th-order of approximation.

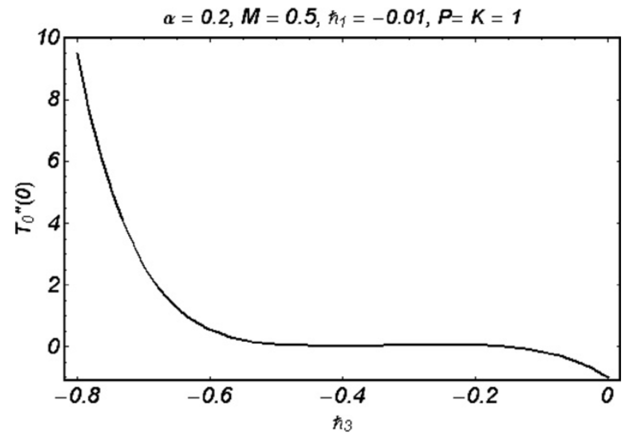

Fig. 3. $\hbar_3$-curve for the 8th-order of approximation.

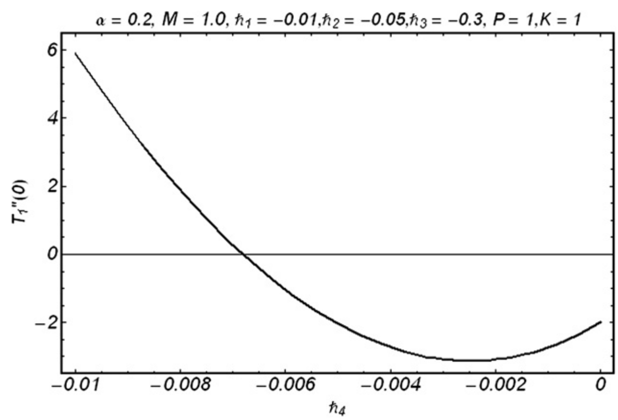

Fig. 4. $\hbar_4$-curve for the 8th-order of approximation.

$-0.6 \leqslant \hbar_3 < 0$ and $-0.004 \leqslant \hbar_4 < 0$. Our calculations clearly indicate that the series (60), (61), (104) and (106) converge for whole region of η when $\hbar_1 = -0.01$, $\hbar_1 = -0.05$, $\hbar_3 = -0.3$ and $\hbar_4 = -0.002$.

6. Results and discussion

Figs. 5–16 have been drawn to see the effects of the second grade parameter α, Hartman number M, radiation

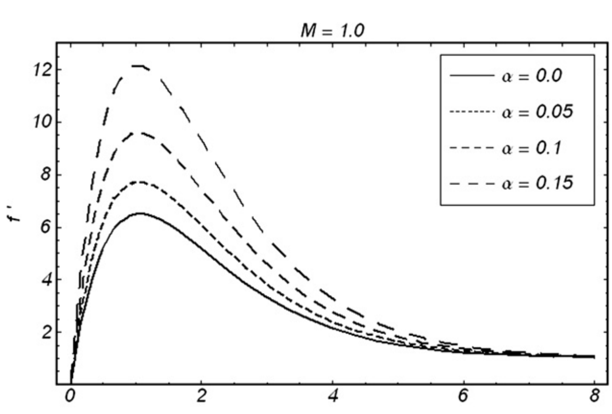

Fig. 5. Effects of α on 8th-order approximation for f' at $\hbar_1 = -0.01$.

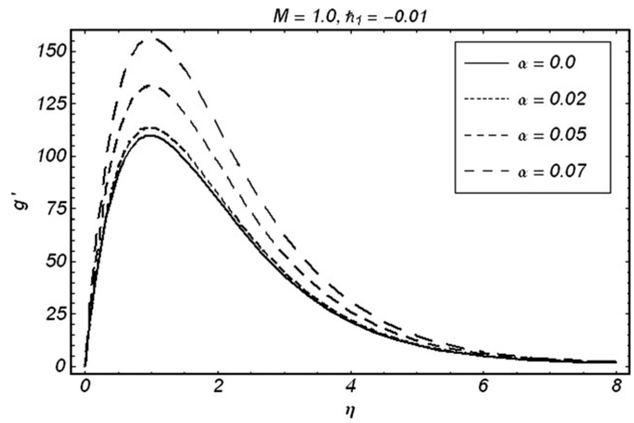

Fig. 6. Effects of α on 8th-order approximation for g' at $\hbar_2 = -0.05$.

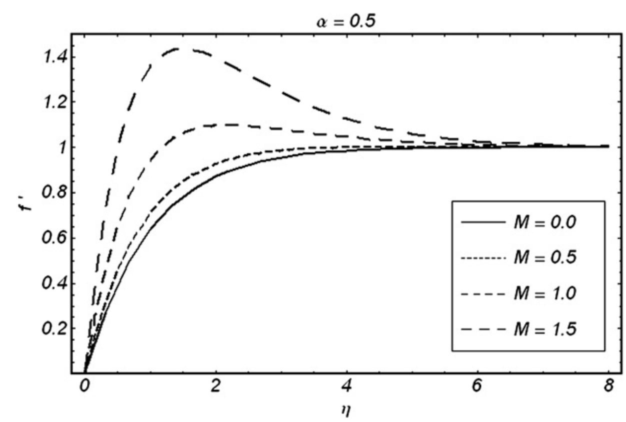

Fig. 7. Effects of M on 8th-order approximation for f' at $\hbar_1 = -0.01$.

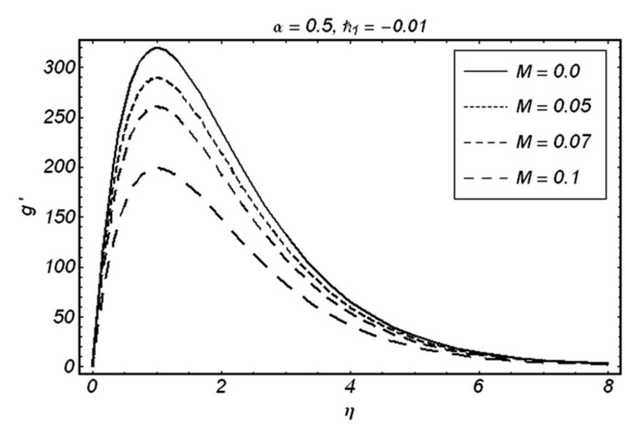

Fig. 8. Effects of M on 8th-order approximation for g' at $\hbar_2 = -0.05$.

parameter K and the Prandtl number Pr on the velocity and the temperature fields.

Figs. 5–8 are made in order to see the effects of α and M on the velocity components f' and g'. From Figs. 5 and 6, it is seen that f' and g' are increased as the second grade parameter α increases but this change is larger in g' when compared with f'. However, the boundary layer thickness decreases in both the cases f' and g'. Figs. 7 and 8 show

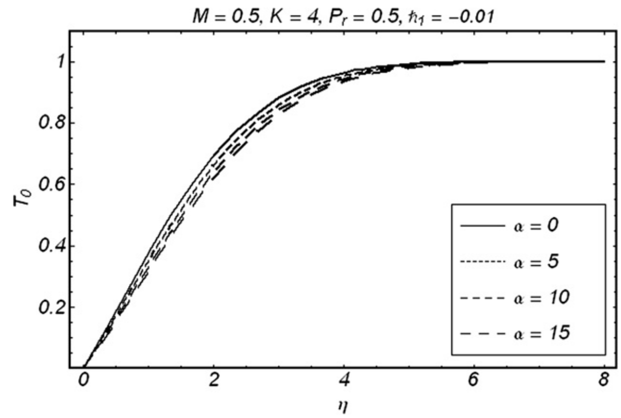

Fig. 9. Effects of α on 8th-order approximation for T_0 at $\hbar_3 = -0.3$.

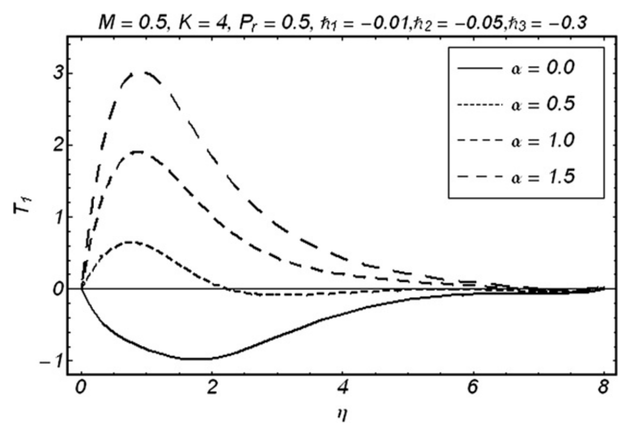

Fig. 10. Effects of α on 8th-order approximation for T_1 at $\hbar_4 = -0.002$.

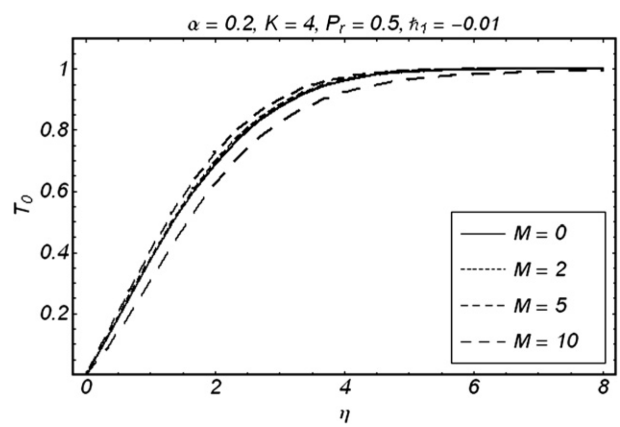

Fig. 11. Effects of M on 8th-order approximation for T_0 at $\hbar_3 = -0.3$.

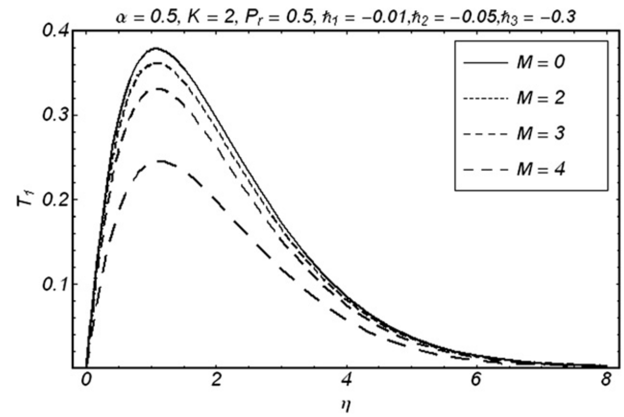

Fig. 12. Effects of M on 8th-order approximation for T_1 at $\hbar_4 = -0.002$.

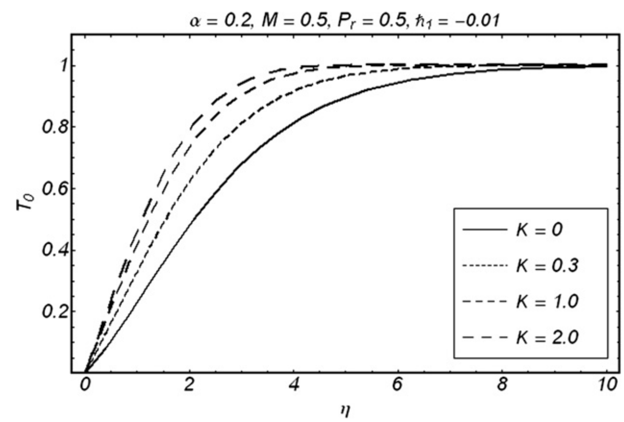

Fig. 13. Effects of K on 8th-order approximation for T_0 at $\hbar_3 = -0.3$.

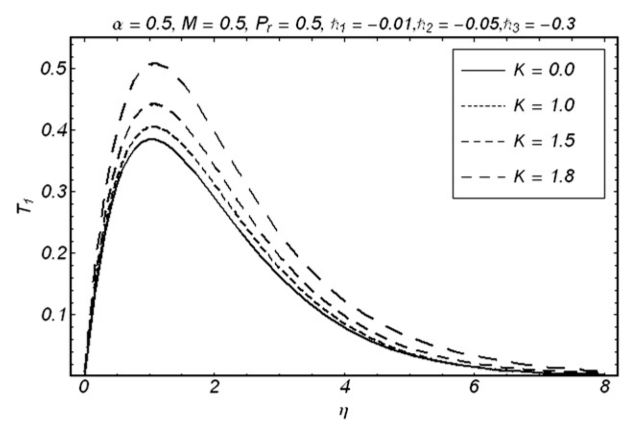

Fig. 14. Effects of K on 8th-order approximation for T_1 at $\hbar_4 = -0.002$.

the effects of M on f' and g'. The velocity f' in Fig. 7 in increasing function of M and the boundary layer thickness decreases in case of f. From Fig. 8, it is seen that the velocity component g' is decreased as the Hartman number M increases. However, this decrement is very larger in g' on the small values of M. The boundary layer thickness increases in this cases.

Figs. 9–16 are plotted to see the effects of α, M, K and Pr on the temperature profiles T_0 and T_1. Figs. 9 and 10 indicate the effects of α on T_0 and T_1. In Fig. 9, the temperature T_0 decreases as α increases but this decrement is made on very large values of second grade parameter α and in Fig. 10, T_1 is increasing when α increases. The boundary layer thickness increases in case of T_0 and decreases in case

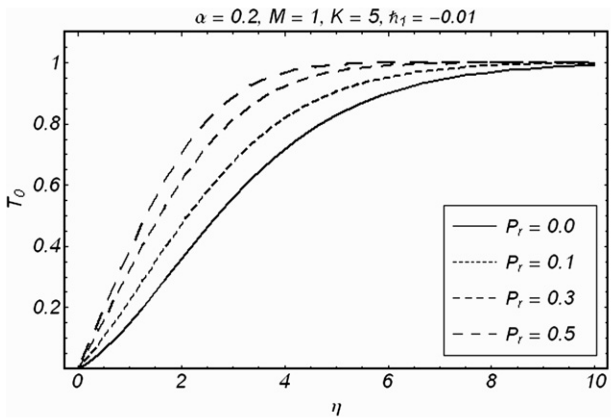

Fig. 15. Effects of Pr on 8th-order approximation for T_0 at $\hbar_3 = -0.3$.

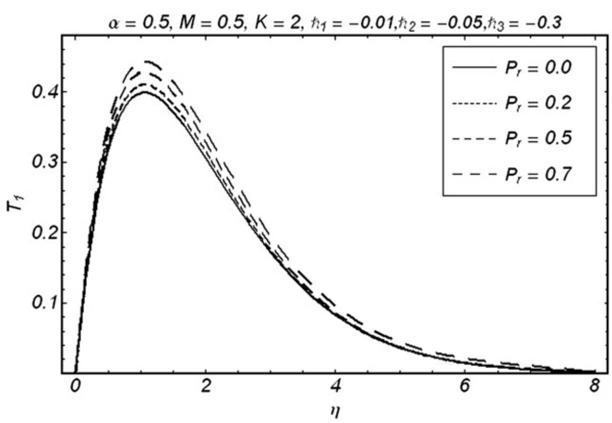

Fig. 16. Effects of Pr on 8th-order approximation for T_1 at $\hbar_4 = -0.002$.

of T_1. Figs. 11 and 12 illustrate the effects of M on temperature T_0 and T_1. It is found in Fig. 11 that T_0 initially increases and after the value at $M \geqslant 10$ it goes to decrease and the boundary layer thickness is increased. Fig. 12 gives that T_1 is decreasing function of M. The boundary layer thickness is also increased in the case of T_1. Figs. 13 and 14 elucidate the effects of K on T_0 and T_1. It is seen that T_0 and T_1 are increasing function of K. Moreover the boundary layer thickness decreases in both the cases T_0 and T_1. Figs. 15 and 16 show the effects of Pr on T_0 and T_1. The temperatures T_0 and T_1 increase for large values of Pr. However, this increment is larger in T_0 when compared with T_1 at very small values of Pr. The boundary layer thickness decreases in both cases of T_0 and T_1.

Table 1
Effects of the non-dimensional parameters M and α on $f''(0)$ and $g''(0)$

α	M	$f''(0)$	$g''(0)$	α	M	$f''(0)$	$g''(0)$
1	0	0.93245764	0.87630411	0	1	1.1329356	2.0831693
	0.1	0.933154	0.877628	0.1		1.11856	1.91021
	0.5	0.949745	0.908814	0.5		1.06353	1.38704
	1.0	1.0000000	1.0000000	1.0		1.0000000	1.0000000
	1.5	1.07884	1.13584	1.5		0.941841	0.783586
	3.0	1.4266105	1.5686494	1.7		0.919979	0.724826
	4.0	1.7023142	−21.505764	2.0		0.88860210	0.65521489

Table 2
Effects of the non-dimensional parameters M, α, K and Pr on $T'_0(0)$ and $T'_1(0)$

α	M	K	Pr	$T'_0(0)$	$T'_1(0)$
1	0	3	1	0.36170565	1.44329
	0.1			0.361766	1.44309
	0.5			0.363197	1.4384
	1.0			0.36751389	1.42398
	1.5			0.374217	1.40059
	3.0			0.40228349	1.27357
	4.0			0.42214090	1.09502
1	2	0	1	0.057648010	1.0000000
		0.1		0.155971	1.00382
		0.5		0.31501	1.02045
		1.0		0.38762665	1.03878
		1.5		0.421388	1.05697
		2.0		0.43071212	1.09287
		3.0		0.38268495	1.36861
		5.0		0.10071477	4.68049
1	1	3	0	0.057648010	1.0000000
			0.1	0.152492	1.01245
			0.3	0.268866	1.03823
			0.5	0.342648	1.06497
			0.7	0.396276	1.09249
			1.0	0.36751389	1.13512
			1.2	−0.205072	1.16406
0	1	3	1	0.38138688	1.12466
0.1				0.379944	1.15557
0.5				0.374298	1.27704
1.0				0.36751389	1.42398
1.5				0.361019	1.56544
1.7				0.358499	1.62049
2.0				0.35479890	1.70141

Tables 1 and 2 have been included just to show the variation of α, M, K, and Pr. Table 1 indicates the effects of α and M on $f''(0)$ and $g''(0)$. It is found that $f''(0)$ increases with an increase in M but decreases for large values of α [45]. The behavior of α and M on the magnitude of $g''(0)$ is similar to that of $f''(0)$. Table 2 shows the variation of M, K, Pr and α on $T'_0(0)$ and $T'_1(0)$. As expected $T'_0(0)$ increases and $T'_1(0)$ decreases when values of M are increased. Further $T'_1(0)$ is an increasing function of K whereas $T'_0(0)$ increases for $0 \leqslant K \leqslant 2.0$ and then decreases when $K \geqslant 2$. The effects of Pr here indicates that $T'_0(0)$ first increases and then decreases. However $T'_1(0)$ is an increasing function of Pr. Moreover, the large values of α are responsible for increasing $T'_1(0)$ and decreasing $T'_0(0)$.

7. Concluding remarks

MHD steady flow of an incompressible second grade fluid in the presence of radiation has been examined in this treatise. The system is stressed by a uniform transverse magnetic field. The non-linear equations are solved analytically using HAM. The obtained results are quite new and have never been reported. Even such results for MHD viscous fluid have not be reported yet. However, the numerical solution for MHD viscous fluid is given in reference

[21]. The results for MHD viscous fluid in the presence of a radiation can be recovered by taking $\alpha = 0$. The present solution for $f''(0)$ and $T_0'(0)$ are qualitatively similar to that given in reference [45]. This provides useful comparison.

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References

[1] K.J. Mysels, Flow of thickened fluids, US Patent No. 2 492 (1949) 173.
[2] B.A. Toms, Some observations on the flow of linear polymer solutions through straight tubes at large Reynolds numbersProceeding of the First International Congress on Rheology, vol. 2, North Holland, Amsterdam, 1948, pp. 135–141.
[3] K.R. Rajagopal, A note on unsteady unidirectional flows of a non-Newtonian fluid, Int. J. Non-Linear Mech. 17 (1982) 369–373.
[4] K.R. Rajagopal, On the creeping flow of a second grade fluid, J. Non-Newtonian Fluid Mech. 48 (1984) 239–246.
[5] T. Hayat, S. Asghar, A.M. Siddiqui, Some non-steady flows of a non-Newtonian fluid, Int. J. Eng. Sci. 38 (2000) 337–346.
[6] A.M. Siddiqui, M.R. Mohyuddin, T. Hayat, S. Asghar, Some more inverse solutions for steady flows of a second grade fluid, Arch. Mech. 55 (4) (2003) 373–387.
[7] W.C. Tan, M.Y. Xu, The impulsive motion of flat plate in generalized second grade fluid, Mech. Res. Comm. 29 (2002) 3–9.
[8] W.C. Tan, M.Y. Xu, Unsteady flows of a generalized second grade fluid with the fractional derivative model between two parallel plates, Acta Mech. Sin. 20 (5) (2004) 471–476.
[9] W.C. Tan, T. Masuoka, Stokes first problem for second grade fluid in a porous half space, Int. J. Non-Linear Mech. 40 (2005) 515–522.
[10] C. Fetecau, C. Fetecau, Starting solutions for some unsteady unidirectional flows of a second grade fluid, Int. J. Eng. Sci. 43 (2005) 781–789.
[11] C. Fetecau, J. Zierep, On a class of exact solutions of the equations of motion of a second grade fluid, Acta Mech. 150 (2001) 135–138.
[12] C. Fetecau, C. Fetecau, J. Zierep, Decay of a potential vortex and propagation of a heat wave in a second grade fluid, Int. J. Non-Linear Mech. 37 (2002) 1051–1056.
[13] A.R. Bestman, S.K. Adjepong, Unsteady hydromagnetic free-convention flow with radiative heat transfer in a rotating fluid, Astrophys. Space Sci. 143 (1988) 73–80.
[14] H.S. Takhar, R. Gorla, V.M. Soundalgekar, Radiation effects on MHD free convection flow of a gas past a semi-infinite vertical plate, Int. J. Numer. Meth. Heat Fluid Flow 6 (1996) 77–83.
[15] M.M. Ali, T.S. Chen, B.F. Armaly, Natural convection-radiation interaction in boundary layer flow over horizontal surfaces, AIAA J. 22 (1984) 1797–1803.
[16] F.S. Ibrahim, Mixed convection-radiation interaction in boundary layer flow over horizontal surfaces, Astrophys. Space Sci. 168 (1990) 263–276.
[17] M.A. Mansour, Radiative and free-convection effects on the oscillatory flow past a vertical plate, Astrophys. Space Sci. 166 (1990) 269–275.
[18] M.A. Hossain, M.A. Alim, D. Rees, The effect of radiation on free convection from a porous vertical plate, Int. J. Heat Mass Transfer 42 (1999) 181–191.
[19] M.A. Hussain, K. Khanafer, K. Vafai, The effects of radiation on free convection flow of fluid with variable viscosity from a porous vertical plate, Int. J. Thermal Sci. 40 (2001) 115–124.
[20] E.M.A. Elbashbeshy, M.F. Dimian, Effects of radiation on the flow and heat transfer over a wedge with variable viscosity, Appl. Math. Comput. 132 (2002) 445–454.
[21] A. Raptis, C. Perdikis, H.S. Takhar, Effects of thermal radiation on MHD flow, Appl. Math. Comp. 153 (2004) 645–649.
[22] S.J. Liao, Beyond Perturbation: Introduction to Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, 2003.
[23] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499–513.
[24] K. Vajravelu, T. Roper, Flow and heat transfer in a second grade fluid over a stretching sheet, Int. J. Non-Linear Mech. 34 (1999) 1031–1036.
[25] S.J. Liao, A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate, J. Fluid Mech. 385 (1999) 101–128.
[26] S.J. Liao, A. Campo, Analytic solutions of the temperature distribution in Blasius viscous flow problems, J. Fluid Mech. 453 (2002) 411–425.
[27] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J. Fluid Mech. 488 (2003) 189–212.
[28] S.J. Liao, K.F. Cheung, Homotopy analysis of nonlinear progressive waves in deep water, J. Eng. Math. 45 (2003) 105–116.
[29] S.J. Liao, I. Pop, Explicit analytic solution for similarity boundary layer equations, Int. J. Heat Mass Transfer 46 (2004) 1855–1860.
[30] M. Ayub, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, Int. J. Eng. Sci. 41 (2003) 2091–2103.
[31] T. Hayat, M. Khan, M. Ayub, On the explicit analytic solutions of an Oldroyd 6-constant fluid, Int. J. Eng. Sci. 42 (2004) 123–135.
[32] T. Hayat, M. Khan, M. Ayub, Couette and Poiseuille flows of an Oldroyd 6-constant fluid with magnetic field, J. Math. Anal. Appl. 298 (2004) 225–244.
[33] T. Hayat, M. Khan, S. Asghar, Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid, Acta Mech. 168 (2004) 213–232.
[34] C. Yang, S.J. Liao, On the explicit purely analytic solution of Von Karman swirling viscous flow, Comm. Non-linear Sci. Numer. Simm. 11 (2006) 83–93.
[35] S.J. Liao, A new branch of solutions of boundary-layer flows over an impermeable stretched plate, Int. J. Heat Mass Transfer 48 (2005) 2529–2539.
[36] S.J. Liao, An analytic solution of unsteady boundary-layer flows caused by an impulsively stretching plate, Comm. Non-linear Sci. Numer. Simm. 11 (2006) 326–339.
[37] J. Cheng, S.J. Liao, I. Pop, Analytic series solution for unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium, Transport Porous Media 61 (2005) 365–379.
[38] H. Xu, S.J. Liao, Series solutions of unsteady magnetohydrodynamic flows of non-Newtonian fluids caused by an impulsively stretching plate, J. Non-Newtonian Fluid Mech. 129 (2005) 46–55.
[39] H. Xu, An explicit analytic solution for convective heat transfer in an electrically conducting fluid at a stretching surface with uniform free stream, Int. J. Eng. Sci. 43 (2005) 859–874.
[40] T. Hayat, M. Khan, Homotopy solution for a generalized second grade fluid past a porous plate, Non-Linear Dynamics 42 (2005) 395–405.
[41] W. Wu, S.J. Liao, Solving solitary waves with discontinuity by means of the homotopy analysis method, Chaos, Solitons & Fractals 26 (2005) 177–185.
[42] Y.Y. Wu, C. Wang, S.J. Liao, Solving the one loop solution of the vakhnenko equation by means of the homotopy analysis method, Chaos, Solitons & Fractals 23 (2005) 1733–1740.
[43] T. Hayat, M. Khan, S. Asghar, Magnetohydrodynamic flow of an Oldroyd 6-constant fluid, Appl. Math. Comput. 155 (2004) 417–425.
[44] M. Sajid, T. Hayat, S. Asghar, On the analytic solution of the steady flow of a fourth grade fluid, Phys. Lett. A 355 (2006) 18–26.
[45] R. Cortell, Effects of viscous dissipation and work done by deformation on the MHD flow and heat transfer of a viscoelastic fluid over a stretching sheet, Phys. Lett. A 357 (2006) 298–305.